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Soliton solutions of driven nonlinear Schrödinger equation

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Abstract
We analyse the structure of the exact, dark and bright soliton solutions of the driven nonlinear Schrödinger equation. A wide class of solutions, phase locked with the source, is identified for distinct parameter ranges. These contain periodic as well as localized solutions, which can be singular implying extreme increase in intensity. Conditions for obtaining non-propagating solutions are also found. A special case, where the scale of the soliton emerges as a free parameter, is obtained. We also study the highly restrictive structure of the localized solutions, when the phase and amplitude get coupled. Numerical solutions are obtained for this case, which reveals presence of periodic solutions. Stability analysis is also carried out through the Crank–Nicolson method.

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1. Introduction

The externally driven, nonlinear Schrödinger equation (NLSE) with a source has been investigated in the context of a variety of physical processes. It arises in the problem of Josephson junction [1], charge density waves [2], twin-core optical fibres [3–7], plasma driven by rf fields [8] and a number of other problems [9]. As compared to NLSE, which is an integrable system [10], not much is known about the exact solutions of this equation. Perturbative solutions around the stable soliton solutions of NLSE with a source have been studied earlier. Analysis around constant background and numerical investigations [11–14] have revealed the phenomenon of auto-resonance [15, 16] as a key characteristic of this system, where a continuous phase locking between the solutions of NLSE and the driven field is observed.

In a recent work [17], some of the present authors have devised a procedure based on fractional linear transformation for obtaining exact solutions of this dynamical system. One obtains both localized and oscillatory solutions. Apart from these regular solutions, under certain constraints singular solutions have also been found, implying extreme increase in the field intensity due to self-focussing. This approach is non-perturbative in the sense that the obtained exact solutions are necessarily of rational type, with both numerator and denominator containing terms quadratic in elliptic functions. However, the exact parameter ranges in which the general solution exhibits bright and dark nature have not been investigated. Considering the importance of the localized solutions of this physically important dynamical system, the above point needs a systematic study. Study of the stability of these as well as the periodic solutions also need careful analysis.

The goal of the present paper is to analyse in detail the structure of the most general, bright and dark solitons of the NLSE with a source. The parametric restrictions, under which singular structures can form, are obtained. Unlike NLSE, it is observed that dark and bright solitons depend both on coupling and source strengths. For example, bright solitons can also form in the repulsive regime, if the source strength has positive value. Similarly dark solitons can form in the attractive regime. Conditions which give non-propagating solutions, for driven NLSE, are studied. The possibility where the phase and amplitude can get related is also investigated. The highly restrictive nature of the resulting dynamics is pointed out. The stability of the solutions has been studied numerically through the Crank–Nicolson finite difference method.

2. Analysis of NLSE phase locked with source

The equation which we intend to solve is NLSE driven with a plane wave, and phase locked with it:

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi + \mu\psi = \kappa\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{1}$$

where g, μ, κ, ω and k are real constants. We consider the ansatz travelling wave solution in the form,

$$\psi(x,t) = \rho(\xi)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$

where $\xi = \alpha(x - vt)$. Separating the real and imaginary parts of equation (1), one obtains,

$$v = 2k, \tag{2}$$

from the imaginary part, indicating that in the present case, the wave velocity v is controlled by k. The real part simply yields,

$$\alpha^2\rho'' + g\rho^3 + \epsilon\rho = \kappa, \tag{3}$$

where $\epsilon = \omega - k^2 + \mu$, and the prime indicates differentiation with respect to ξ. It has been observed earlier [17] that this equation can be connected to the equation $f'' + af + bf^3 = 0$ through the following fractional linear transformation (FT):

$$\rho(\xi) = \frac{A + Bf(\xi|m)^\delta}{1 + Df(\xi|m)^\delta} \tag{4}$$

where A, B and D are real constants, δ is an integer and $f(\xi|m)$ is a Jacobi elliptic function, with the modulus parameter m. It can be shown that $\delta = 2$ is the maximum allowed value for which $E_0 = \frac{f'^2}{2} + \frac{b}{4}f^4 + \frac{a}{2}f^2$ is conserved.

We consider the case where $f(\xi|m) = \mathrm{cn}(\xi|m)$; other cases can be similarly studied. Since the goal is to study the localized solutions systematically, we consider the case with modulus parameter $m = 1$, which reduces $\mathrm{cn}(\xi)$ to $\mathrm{sech}(\xi)$. It is worth pointing out that other solutions involving $\mathrm{sn}(\xi)$ and $\mathrm{dn}(\xi)$ naturally emerge from the above solution, since the transform involves square of the $\mathrm{cn}(\xi)$ function.

We can see that equation (4) connects $\rho(\xi)$ to the Jacobi elliptic equation, provided $AD \neq B$, and the following conditions are satisfied for the localized solutions:

$$A\epsilon + gA^3 - \kappa = 0, \tag{5}$$

$$2\epsilon AD + \epsilon B + 4\alpha^2(B - AD) + 3gA^2B - 3\kappa D = 0, \tag{6}$$

$$A\epsilon D^2 + 2\epsilon BD + 4\alpha^2(AD - B)D + 6\alpha^2(AD - B) + 3gAB^2 - 3\kappa D^2 = 0, \tag{7}$$

$$\epsilon BD^2 + 2\alpha^2(B - AD)D + gB^3 - \kappa D^3 = 0. \tag{8}$$

Equation (5) in A does not involve B and D, which is first solved to get the real A. Thus A is determined in terms of ϵ, κ and g. From equation (6), we determine D in terms of B as $D = \Gamma B$, where $\Gamma = \frac{\epsilon+4\alpha^2+3gA^2}{4\alpha^2A+3\kappa-2\epsilon A}$. By substituting this into equation (7), B is found as $B = \frac{6\alpha^2(1-A\Gamma)}{3gA+A\epsilon\Gamma^2+2\epsilon\Gamma+4\alpha^2\Gamma(A\Gamma-1)-3\kappa\Gamma^2}$. From equation (8), we obtain a cubic equation in $\beta \equiv \alpha^2$:

$$p_1\beta^3 + q_1\beta^2 + r_1\beta + c_1 = 0, \tag{9}$$

where $p_1 = 64(A^3g + A\epsilon - \kappa)$, $q_1 = (48A^5g^2 + 64A^3g\epsilon + 16A\epsilon^2 - 48A^2g\kappa - 16\epsilon\kappa)$, $r_1 = (12A^7g^3 + 36A^5g^2\epsilon + 20A^3g\epsilon^2 - 4A\epsilon^3 - 60A^4g^2\kappa - 72A^2g\epsilon\kappa + 4\epsilon^2\kappa + 48Ag\kappa^2)$ and $c_1 = (3A^7g^3\epsilon - 3A^5g^2\epsilon^2 - 7A^3g\epsilon^3 - A\epsilon^4 - 18A^6g^3\kappa - 15A^4g^2\epsilon\kappa + 12A^2g\epsilon^2\kappa + \epsilon^3\kappa + 9A^3g^2\kappa^2 - 15Ag\epsilon\kappa^2 + 9g\kappa^3)$. It can be straightforwardly seen that p_1 in equation (9) is the consistency condition (5) and hence is identically zero. Therefore, the width parameter β is the solution of a quadratic equation. Thus for any given values of g, ϵ and κ, we can find the values of A, B, D and α.

So, the localized solutions are of the form

$$\psi = \frac{A + B\,\mathrm{sech}(\xi)^2}{1 + D\,\mathrm{sech}(\xi)^2}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{10}$$

where the constants obey relations (5)–(8). The expressions for A, B, D and α in terms of g, κ and ϵ are lengthy and are not particularly illuminating. Hence, the explicit expressions are not given here; they are computed from the given values of g, κ and ϵ. We have checked that solutions of the type

$$\psi = \frac{A + B\mathrm{cn}(\xi)}{1 + D\mathrm{cn}(\xi)}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \tag{11}$$

are also solutions of the NLSE with source, for appropriate values of A, B, D and α. The corresponding localized solutions ($m = 1$) are of the type obtained earlier [11].

3. Analysis of the localized solutions

We now proceed to analyse carefully the localized solutions of equation (3). It is clear that a localized solution is bound to have at least one extremum in its profile. And since the first derivative of $\rho(\xi)$ must vanish at the extremum, therefore,

$$\frac{2(B - AD)f(\xi)}{(1 + Df(\xi)^2)^2}f'(\xi) = 0. \tag{12}$$

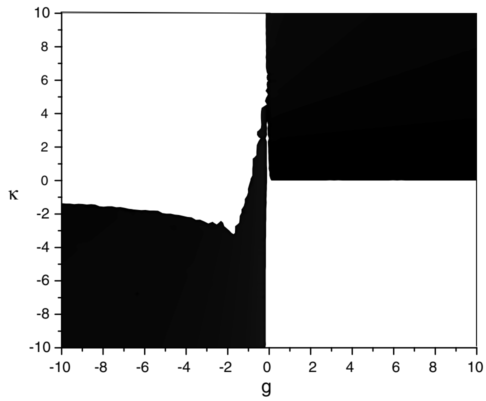

Figure 1. Contour plot depicting positive and negative values of A for different values of g and κ; here $\epsilon = 0.1$. The dark region shows the values of g and κ where A takes positive values, whereas the white region corresponds to negative values of A.

Since $AD \neq B$, either f or f' or both must be zero. But we consider $f(\xi) = \mathrm{sech}(\xi)$, whose first derivative vanishes only at origin. This means we have an extremum at origin. The second derivative of ρ at origin is,

$$\rho'' = \frac{2(AD - B)}{(1 + D)^2}, \tag{13}$$

which resolves the maximum and minimum. It should be noted that ρ'' is singular for $D = -1$. For the non-singular case, we see that there is a clear distinction of two regimes of solutions: one for which $AD > B$, where ρ'' is positive; this corresponds to *minimum*. In the second case, where $AD < B$, we have a *maximum*. The latter corresponds to a *bright soliton*, whereas the former corresponds to a *dark soliton* or *background soliton* in the propagating media. This clearly suggests that both types of solitons exist in this dynamical system.

In these rational solutions, parameter A decides the strength of the background, in which these solutions propagate. It is interesting to note that for the localized solution, $A = 0$ is not permitted, since this leads to the absence of the source. Considering ρ to be positive semi-definite one finds further constraints: (i) A should be positive; (ii) $A > |B|$; (iii) D should be greater than -1 for non-singularity. The case of negative ρ can be tackled analogously; we concentrate on the positive case below. The parameters satisfying the above conditions are taken to be physically meaningful. The parameter conditions leading to singular solutions are dealt with separately. Figure 1 shows the allowed parameter values for the localized solutions considered above. Figure 2 shows the solutions for some mentioned values of the parameters. These values have been used in the relation (5)–(8) to compute A, B, D and α which characterize the solution.

Consider the case when the source has no space dependence, that is, when $k = 0$, which in the light of equation (2) gives propagation velocity of the envelope as zero. So, in this case the solitons are stationary and non-propagating ones; however, the profile of the solution remains unchanged, as k affects the solution implicitly via ϵ.

For the case when $\epsilon = 0$ for which $\omega = k^2 - \mu$; in this case equations (5)–(8) yield

$$q_1(\alpha^4) + r_1\alpha^2 + c = 0. \tag{14}$$

Here, $q_1 = 48A^5g^2 - 48A^2g\kappa$, $r_1 = 12A^7g^3 - 60A^4g2\kappa + 48Ag\kappa^2$ and $c = -18A^6g^3\kappa + 9A^3g^2\kappa^2 + 9g\kappa^3$. Very interestingly, all of these coefficients vanish in view of equation (5),

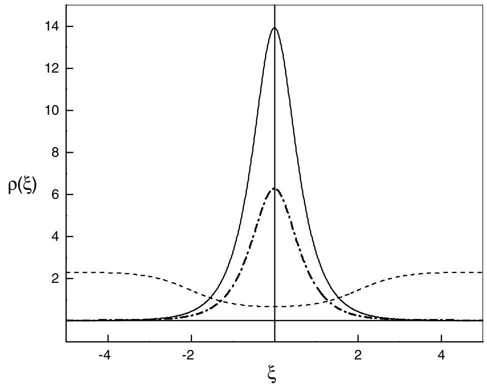

Figure 2. Density profiles of some solitons of driven NLSE. (i) Dark soliton (dashed), with $g = 0.7, \kappa = -0.7$ and $\epsilon = -1.4$; (ii) bright soliton (solid), with $g = -0.5, \kappa = 0.1$ and $\epsilon = -0.4$; (iii) bright soliton (dash-dotted), with $g = -0.4, \kappa = 0.1$ and $\epsilon = -0.4$; A, B, D and α which characterize the solution have been computed from these values.

leaving α as a free parameter. So, the width of the soliton, in this case, is independent of the parameter values, which means that the solitons can have arbitrary size for the given values of g and κ.

As noted earlier, equation (3) allows singular solutions for $D = -1$. One can show that they satisfy the relation

$$g(6\alpha^2 - 3\epsilon)A^2 + \epsilon(2\alpha^2 + \epsilon) + 9g\kappa A - 8\alpha^4 = 0. \tag{15}$$

These solutions possess one singularity in their profile, which physically corresponds to very large field intensity due to self-focussing [18, 19]. When $D < -1$, singular solitons exist, with singularities existing at two different locations, between which ρ becomes negative.

Stability of the wide variety of solutions found here needs to be investigated. Stability of the constant background and inhomogeneous solutions for NLSE with source has been investigated earlier [11]. It has been observed that in the presence of damping this dynamical system possesses a rich structure, involving stable solutions, spatio-temporal chaos and unstable regimes. In the absence of damping, for weak source strength, stable solutions had been identified. We have studied the stability of solutions numerically using the Crank–Nicolson method, which is unconditionally stable, and found a number of solutions to be stable in agreement with the results of Barashenkov *et al* [11]. Here we show the stability of localized, non-propagating solutions. Figure 4 depicts the magnitude of perturbed solutions with the initial condition $\psi(x, t = 0) = \psi(x, t = 0) + \epsilon$, where ϵ is a function which assumes a random value at each point, where the maximum value of ϵ is 10% of the peak value of ψ. One can clearly see from figure 3, which is the time evolution of the magnitude of unperturbed solution, that perturbation does not destabilize the solution. We have checked addition of random noise up to 30% of the peak value of ψ; the localized solutions are found to be quite stable. The step sizes dx and dt were taken as 0.01 and 0.0001, respectively, in this method.

4. NLSE with a generalized source

Considering a more generalized source, rather than a plane wave source, we have the driven NLSE as

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi + \mu\psi = \kappa\,\mathrm{e}^{\mathrm{i}(\chi(\xi)-\omega t)}. \tag{16}$$

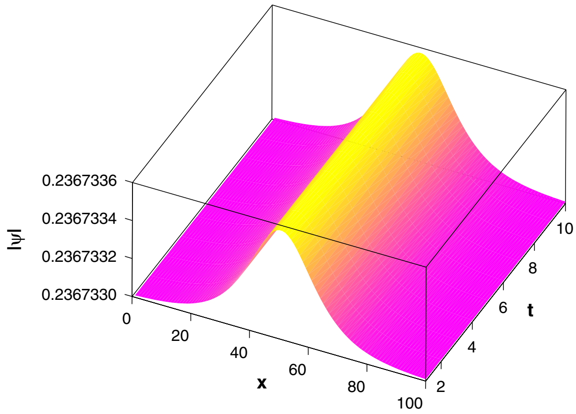

Figure 3. Time evolution of the unperturbed solution.

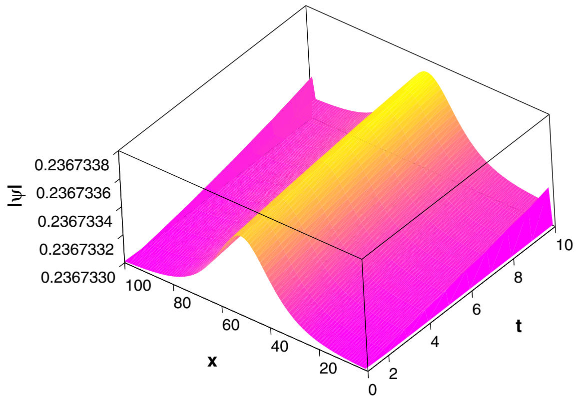

Figure 4. Time evolution of the noise perturbed solution, the stability of the solution is clearly seen.

Here $\chi(\xi)$ is some function of x and t. We consider the ansatz,

$$\psi(x,t) = \rho(\xi,t)\,\mathrm{e}^{\mathrm{i}(\chi(\xi)-\omega t)}, \tag{17}$$

where $\xi = \alpha(x - vt)$. Equation (17) when substituted into equation (16) gives a complex equation in ρ and χ. Equating the imaginary part to zero, one gets

$$\chi' = \frac{v}{2\alpha} + \frac{c}{\rho^2}. \tag{18}$$

One can clearly see that the above equation suggests phase-amplitude coupling for $c \neq 0$. We see from relation (18) an interesting case of the phase singularity of the soliton arising through phase-amplitude coupling, when ρ vanishes.

Substituting this relation into the real part of equation (16), we get a nonlinear differential equation of the form

$$\alpha^2\rho'' + g\rho^3 + \epsilon\rho = \kappa + \frac{\alpha^2 c^2}{\rho^3}, \tag{19}$$

where $\epsilon = \left(\omega + \mu + \frac{v^2}{4}\right)$. The above equation can be connected with the equation governing Jacobi elliptic functions via a fractional transformation as

$$\rho(\xi) = \frac{A + Bf(\xi|m)^\delta}{1 + Df(\xi|m)^\delta}, \tag{20}$$

where $\delta = 2$ is again the maximum allowed value. This gives the consistency conditions:

$$gA^6 + \epsilon A^4 - (2\alpha^2(AD - B)(1 - m) + \kappa)A^3 = \alpha^2 c^2 \tag{21}$$

$$\begin{aligned} &6A^5Bg + 4\epsilon A^3B + 2\epsilon A^4D - 3\kappa A^2B - 3\kappa A^3D + 6\alpha^2(B - AD)(1 - m)A^2B \\ &\quad + 6\alpha^2(AD - B)(1 - m)A^3D + 4\alpha^2(2m - 1)A^3(B - AD) = 6\alpha^2c^2D \end{aligned} \tag{22}$$

$$\begin{aligned} &6\alpha^2(B - AD)(1 - m)AB^2 + 18\alpha^2(AD - B)(1 - m)A^2BD \\ &\quad + 12\alpha^2(B - AD)(2m - 1)A^2B + 4\alpha^2(AD - B)(2m - 1)A^3D \\ &\quad + 6\alpha^2(AD - B)mA^3 + 15gA^4B^2 + 6\epsilon A^2B^2 + 8\epsilon A^3BD + \epsilon A^4D^2 \\ &\quad - 3\kappa AB^2 - 9\kappa A^2BD - 3\kappa A^3D^2 = 15\alpha^2c^2D^2 \end{aligned} \tag{23}$$

$$\begin{aligned} &20gA^3B^3 + 4\epsilon AB^3 + 12\epsilon A^2B^2D + 4\epsilon A^3BD^2 - \kappa B^3 - 9\kappa AB^2D - 9\kappa A^2BD^2 - \kappa A^3D^3 \\ &\quad + 18\alpha^2AB^2D(AD - B)(1 - m) - 2\alpha^2B^3(AD - B)(1 - m) \\ &\quad - 12\alpha^2AB^2(AD - B)(2m - 1) - 2\alpha^2mA^3D(AD - B) \\ &\quad + 12\alpha^2A^2BD(AD - B)(2m - 1) = 20\alpha^2c^2D^3 \end{aligned} \tag{24}$$

$$\begin{aligned} &15gA^2B^4 + \epsilon B^4 + 8\epsilon AB^3D + 6\epsilon A^2B^2D^2 - 3\kappa B^3D - 9\kappa AB^2D^2 - 3\kappa A^2BD^3 \\ &\quad + 6\alpha^2B^3D(AD - B)(1 - m) - 4\alpha^2B^3(AD - B)(2m - 1) \\ &\quad + 12\alpha^2AB^2D(AD - B)(2m - 1) + 18\alpha^2mAB^2(AD - B) \\ &\quad - 6\alpha^2mA^2BD(AD - B) = 15\alpha^2c^2D^4 \end{aligned} \tag{25}$$

$$\begin{aligned} &6gAB^5 + 2\epsilon B^4D + 4\epsilon AB^3D^2 - 3\kappa AB^2D^3 - 3\kappa B^3D^2 + 4\alpha^2(B - AD)(1 - 2m)B^3D \\ &\quad + 6\alpha^2m(B - AD)AB^2D + 6\alpha^2m(AD - B)B^3 = 6\alpha^2c^2D^5 \end{aligned} \tag{26}$$

$$2\alpha^2(B - AD)mB^3D + gB^6 + \epsilon B^4D^2 - \kappa B^3D^3 = \alpha^2c^2D^6. \tag{27}$$

Since there, in total, are seven simultaneous relations to fix three independent parameters, we can see that these are constrained solutions. One can also see that the phase-amplitude coupling imposes constraints on the solution space, without changing the profile of the solutions. From the above relations, one can see that $B = 0$ needs $c = 0$ or $D = 0$. The former case has already been studied here, whereas the latter case results in a constant background solution. Here $A \neq 0$ requires $c \neq 0$; hence the solutions always exist in a constant background. This should be contrasted with the case when $c = 0$; for which cnoidal wave solutions are possible with $A = 0$. Similarly, in $m = 1$ case localized solutions with $B = 0$ are allowed in the previous case, which as seen above are not found in the present case. Hence, only rational solutions are possible. Moreover, we see that (21) is a sixth-order polynomial, in contrast to the previous relations, this does not have analytically tractable roots leaving numerical analysis as the only tool to analyse the structure of solution space apart from some special cases.

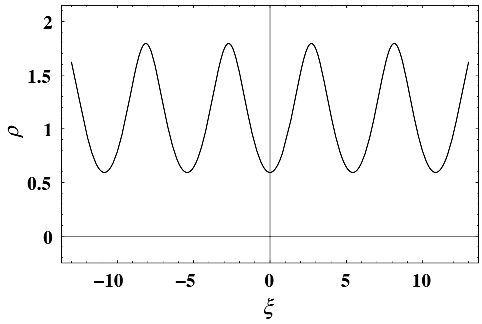

Figure 5. Periodic solutions with $g = 0.875\,976$, $\kappa = -0.478\,279$, $\alpha = -1.438\,06$, $\tilde{c} = 0.186\,253$ and $\epsilon = -1.132\,85$.

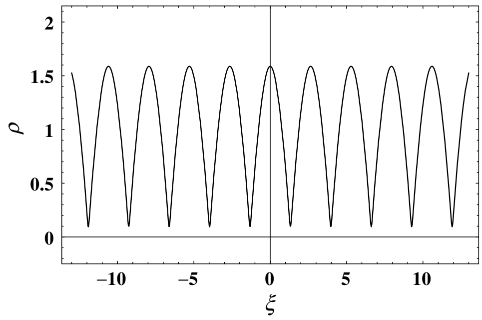

Figure 6. Periodic solutions with $g = 0.449$, $\kappa = 1.7695$, $\alpha = -0.799$, $\tilde{c} = 0.1733$ and $\epsilon = -1.3435$.

Below we illustrate the procedure to solve the consistency conditions with an explicit case when $\epsilon = 0$. Writing $\alpha c = \tilde{c}$ one observes that the relations simplify considerably; equation (21) yields

$$E = \frac{-\kappa \pm \sqrt{\kappa^2 + 4g\tilde{c}^2}}{2g}, \tag{28}$$

where $E = A^3$. It can be clearly seen that A can be both positive and negative; reality of A needs a further constraint on the parameters. Subsequently, assuming $B = \Gamma D$, we have from (22)

$$\Gamma = \frac{6\tilde{c}^2 + 4\alpha^2 A^2 + 3\kappa A^3}{6gA^5 - 3\kappa A^2 - 4\alpha^2 A^2}. \tag{29}$$

Eliminating α^2 from equations (22) and (23), we get

$$D = \frac{18\tilde{c}^2 - 18gA^2\Gamma + 9\kappa A^2\Gamma + 9\kappa A^4}{30gA^4\Gamma^2 + (36g + 12\kappa)A\Gamma^2 - 12\kappa A^3 - 42\tilde{c}^2} \tag{30}$$

which is free from α^2. Using the above relation with equation (27) one finds

$$\alpha^2 = \frac{(g\Gamma^6 - \kappa\Gamma^3 - \tilde{c}^2)(9\tilde{c}^2 - 9gA^2\Gamma + \frac{9}{2}\kappa A^2\Gamma + \frac{9}{2}\kappa A^4)}{(A - \Gamma)\Gamma^3 + 30gA^4\Gamma^2 + (36g + 12\kappa)A\Gamma^2 - 12\kappa A^2 - 42\tilde{c}^2} \tag{31}$$

which can be substituted into (29) to give Γ in terms of g, κ and $\tilde{c}^2$. At this stage, there are three more relations, which in principle can be used to find the values of g, κ and $\tilde{c}^2$ for which the localized solutions exist. Numerical integration of equation (19) yields a variety of solutions for different values of c. We show in figures 5 and 6 a few periodic solutions.

In conclusion, a number of interesting features have emerged from the analysis of the exact solutions of the driven NLSE. Dark and bright solitons can exist in attractive and repulsive nonlinear regimes, a feature very different for NLSE; the presence of the external source makes this possible. Static solitons are found for both the equations. In particular, static solitons are found to have interesting properties like arbitrary scale, under certain parametric restrictions. They are found to be quite stable and robust to noise perturbation. In certain specific parameter regimes, solitons of arbitrary size are found. Singular solutions are also found for both the equations. The investigation of the situation, where the phase is allowed to depend upon the intensity, revealed that the corresponding solutions are highly constrained. The study of solitary waves having complex envelope, analogous to Bloch solitons in condensed matter physics, is worth investigating in the present scenario. Application of the fractional linear transformation technique employed here to other nonlinear equations is also of deep interest. Investigations along these lines are currently in progress.

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